Abstract

This is an introduction to more advanced topics in measurement suitable for an advanced Grade 4 audience. These notes were prepared for the Grand River Chinese School.

This topic is new this year, though it adapts and incorporates content from the geometry unit I taught in previous years. Historically, I taught a four-class geometry primer discussing angles, angles of polygons, and area. Confusion has always arisen with units of measurement, which I suspect is due to the lack of education about units and dimension in school. Hence, I have substantially revised and expanded the curriculum to include much more discussion about dimension and units.

However, the quality of most of these notes is untested, and I may not end up presenting all of the information. I anticipate that each chapter will take around two to four hours to present, depending on the level of detail.

Although the notes are intended to be presented to a young audience, they are written for a teacher and not for a student. Many of the terms used will not be familiar to the students. They require explanation.

Measurement

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Contents

Chapter 1

Introduction

This unit is about geometry and measuring the physical world. Our focus will be on figures on a two-dimensional surface, and the quantities we can use to measure those figures.

1.1 Planes

Often, we like to do geometry on a flat surface. Note that this is not the only kind of geometry; we could do geometry in three dimensions, for example. Many physicists and mathematicians investigate even stranger geometries, such as geometries with time. However, flat geometry is much easier to understand and visualize, and is interesting in its own right. In fact, even if we work in three or more dimensions, we often still study interactions that occur on planes within a higher-dimensional space.

The study of geometry on a flat surface is called plane geometry. The flat surfaces that we do our work on are called planes. For this unit, we will be doing plane geometry.

1.2 Lines

To understand angles, first we must consider straight lines. In geometry, there is a specific definition for what is a line. For example, the object depicted in Figure 1.1 is not a line. Why not? In regular geometry, a line is defined as continuing forever in both directions. Straight objects that terminate at both ends are instead called line segments.

We cannot physically draw an infinitely long line. Instead, we use arrows at both ends to denote that our line continues forever at both ends. Figure 1.2 demonstrates how arrows can be used.

The existence of line segments, which are terminated at both ends, and lines, which continue forever at both ends, raises a question. Are there geometric objects that terminate at one end, and continue forever at the other end? Yes! These kinds of objects are called rays. They behave like lines in one direction, and like line segments in the other direction. Figure 1.3 shows how one might draw a ray.

Figure 1.1: A line segment — not a line!

Figure 1.2: One way to draw a line continuing infinitely in both directions

Figure 1.3: One way to draw a ray terminating at one end and continuing forever in the other direction

A ray has a direction: the end that it terminates at is where the ray starts, and travelling toward its infinite end is considered moving in the ray's direction. You may have heard of a "ray of light". This phrase comes from the well-known behaviour of light: if we shine a flashlight, the resulting ray goes on forever (though perhaps with ever-decreasing intensity) in the direction the light is shown. But there is a clear starting point. Behind the flashlight, there is no light.

Although this could be confusing at first, the point that the ray ends at is both a start and an end. It is a start because the ray begins at that point if we travel along the ray in the direction it points. But it is also an end, because equivalently we could say that the ray ends at that point if we travel along the opposite direction. To minimize the confusion, from now on we will consistently call that point the endpoint of the ray.

1.3 Properties of Lines

There are a few facts about straight lines, line segments, and rays on a plane that are useful to know:

- If two lines are the same line (they occupy the same space), then they are called coincident.
- The straight line segment is the shortest route between two points.
- If two lines never intersect, they are called parallel.
- Two distinct (not coincident) straight lines will never intersect at two or more points.

Example 1a. **Intersecting Lines I**

Three lines are drawn on a plane, no two parallel or coincident. Is it possible for there to be four points on the plane that are on more than one line?

Solution: $\boxed{\text{No}}$. There can only be up to three points, since there is only up to one on both the first and second lines, up to one point on both the first and third lines, and up to one point on both the second and third lines. Therefore, there can only be up to

$$1 + 1 + 1 = 3$$

points in total that are on more than one line.

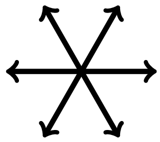

Figure 1.4: Three lines interesting at a single point

Example 1b. **Intersecting Lines II**

Three lines are drawn on a plane, no two parallel or coincident. Is it possible for there to be only one point that is on more than one line?

Solution: $\boxed{\text{Yes}}$. We just require three lines that intersect together at one point. This is basically the design of the asterisk character *; see Figure 1.4.

1.4 Length

It does not make sense to talk about the length of a line or ray, because those objects extend forever. Line segments, on the other hand, do have a well-defined length. In the physical world, length is typically measured in units of kilometres, metres, centimetres, or millimetres.

Length is related to distance. In fact, the distance between two points is the length of the straight line segment connecting those two points. Therefore, distance is also measured in units of length.

It is meaningless to say that the distance between two objects is 3—but it is meaningful to say that the distance is 3 m. The unit is not optional; it is a fundamental component of what the quantity means.

That is not to say, however, that units cannot be changed. Indeed, it is possible to convert between various units of length. The SI system of units in use around the world makes conversions especially convenient. It is useful to remember that:

- $1\,\text{km} = 1000\,\text{m}$
- $1\,\text{m} = 100\,\text{cm}$
- $1\,\text{cm} = 10\,\text{mm}$

Example 1c. **Unit Conversion I**

Convert 1 km to millimetres.

Solution: Note $1\,\text{km} = 1000\,\text{m}$ and $1\,\text{m} = 1000\,\text{mm}$, so

$$1\,\text{km} = 1000 \cdot 1\,\text{m} = 1000 \cdot 1000\,\text{mm} = \boxed{1\,000\,000\,\text{mm}}$$

Chapter 2

Angles

2.1 Angles between Rays

Consider Figure 2.1, which depicts two rays that share a common endpoint. Then look at Figure 2.2, which also depicts two rays with a common endpoint, but further apart. At least, to our eyes the two rays look further apart. But what does it mean for rays to be far apart?

This is no longer a simple matter of distance. Because the rays share a common endpoint, the distance between them is zero. That means that the rays are no more distant from each other in Figure 2.2 than in Figure 2.1. Rather, we need another way to measure how far two rays are. We will use the angle between the rays to describe this.

The size (also called "measure") of an angle is measured in degrees. We use 90° to denote the angle of an L. Therefore, for example, 45° would be half the size of that angle. We will soon develop more intuition into what angles measuring a particular number of degrees look like.

To make it clear that we are talking about the angle between two rays, we can draw figures with part of a circle sandwiched between two rays. This part-circle indicates that the figure's subject is the angle between the rays. Often, we can add additional description near the part-circle, such as a measure of how large the angle is. In Figure 2.3, a 30° angle is depicted.

2.2 Classifying Angles by Measure

2.2.1 Right and Straight Angles

When two rays meet either other in an L shape, the resulting angle is called a right angle. Figure 2.4 depicts a right angle. Note that we often use a square to represent a right angle instead of drawing part of a circle. Right angles are everywhere. For instance, rectangles have four right angles. Take a look around the classroom and note how many right angles you see.

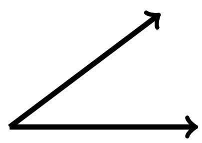

Figure 2.1: Two rays sharing a common endpoint

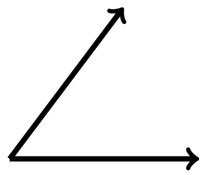

Figure 2.2: Two rays sharing a common endpoint, but further apart

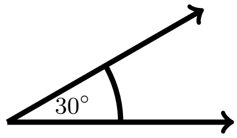

Figure 2.3: An angle measuring 30° with a part-circle highlighting it

Another special kind of angle occurs when two rays of opposite directions meet at a point. Since the rays together form a straight line, such angles are called straight angles. Figure 2.5 depicts a straight angle.

2.2.2 Acute and Obtuse Angles

The right angle, as mentioned earlier, is given a size of 90°. Right angles are so important that other angles are classified based off their size in relation to the right angle. Angles measuring more than 0°, but less than 90°, are classified as acute angles. Angles measuring more than 90°, but less than 180°, are classified as obtuse angles.

Example 2a. **Acute, Right, Obtuse**

Circle the correct category for each angle.

- 41° Acute Right Obtuse
- 175° Acute Right Obtuse

Solution: Because $0° < 41° < 90°$, 41° is an $\boxed{\text{acute}}$ angle. Because $90° < 175° < 180°$, 175° is an $\boxed{\text{obtuse}}$ angle.

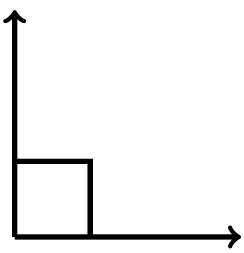

Figure 2.4: Two rays meeting at a right angle, with the right angle labelled using a square shape

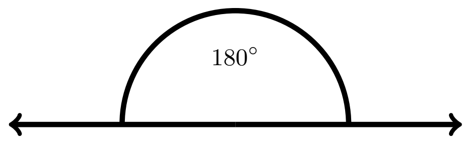

Figure 2.5: Two rays meeting at a straight angle

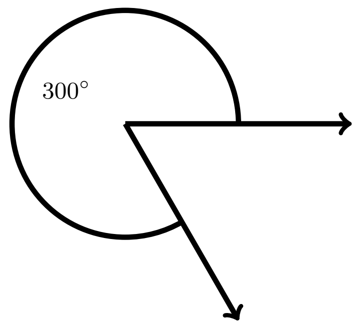

Figure 2.6: Two rays meeting at an acute angle, but the reflex angle is highlighted

2.2.3 Reflex Angles

When two rays intersect, there are actually two angles that one can draw. One of these angles is an acute, right, obtuse, or straight angle; the other is either a straight angle or a so called reflex angle. Reflex angles are greater than 180° but less than 360°. A reflex angle is depicted in Figure 2.6; note that it combines with an acute angle to form the full circle.

Reflex angles are not as important as acute, right, and obtuse angles; therefore we will not focus on them much for the remainder of this course.

2.3 Angle Arithmetic

We can add two angles placed next to each other together, forming a bigger angle. The measure of this bigger angle is the sum of the measures of the two components. In Figure 2.7, we see two acute angles measuring 30° and 60°, and their sum is 90°. We can add degrees just by adding the numbers together and leaving the unit as-is.

Similarly, we can subtract two angles if one is contained within the other. The angle left over is a smaller angle. In Figure 2.8, we see a 60° angle taken away from a 120° angle, leaving another 60° angle as the difference.

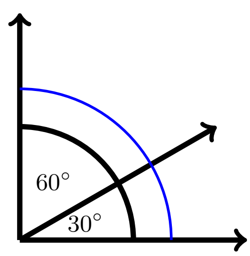

Figure 2.7: Two acute angles summing to a right angle; the sum is highlighted in blue

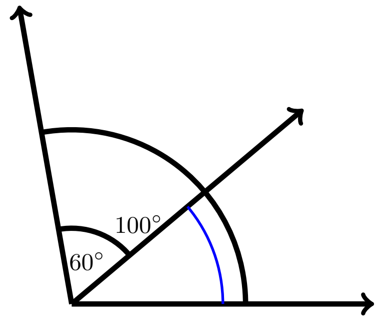

Figure 2.8: An acute angle is subtracted off an obtuse angle, leaving an acute angle; the difference is highlighted in blue

Example 2b. **Adding Angles I**

Find each sum or difference of angles.

- $20° + 30°$
- $155° - 65°$

Solution: We have

$$20° + 30° = (20 + 30)° = \boxed{50°}$$

And

$$155° + 65° = (155 - 65)° = \boxed{90°}$$

Example 2c. **Adding Angles II**

Determine whether each case is possible. If so, give an example. Otherwise, give a reason why it is not possible.

- The sum of two acute angles is an acute angle.
- An obtuse angle minus a right angle is an obtuse angle.

Solution: The first case is $\boxed{\text{possible}}$; we take for example

$$30° + 40° = 70°$$

which is acute.

The second case is $\boxed{\text{not possible}}$. No obtuse angle is 180° or larger; therefore, subtracting 90° will always result in an angle less than 90°—an acute angle.

2.3.1 Complementary and Supplementary Angles

If two angles labelled α and β sum to $90°$, then we say that α and β are complementary angles. If two angles labelled α and β sum to $180°$, then we say that α and β are supplementary angles. You will not be required to memorize these terms; the quiz and homework will remind you of what they mean.

Example 2d. **Complementary Angles**

Find the angle complementary to each of the following.

- $81°$ — Complementary: ______°
- $18°$ — Complementary: ______°

Solution: We just calculate

$$90° - 81° = \boxed{9°}$$

and

$$90° - 18° = \boxed{72°}$$

Example 2e. **Supplementary Angles**

Find the angle supplementary to each of the following.

- $81°$ — Supplementary: ______°
- $18°$ — Supplementary: ______°

Solution: We just calculate

$$180° - 81° = \boxed{99°}$$

and

$$180° - 18° = \boxed{162°}$$

2.4 Angles as Turns

Imagine rotating an object around some pivot point. How can we measure how much an object has rotated? One way would be to measure the number of full rotations, defining a full rotation to be a complete turn around the pivot. After a full rotation, the object is in the same orientation that it was in originally.

Consider a straight bar with both tips coloured differently, initially oriented horizontally. We draw the initial position here and define this position to be the starting position.

TK.

2.5 Angles of a Polygon

TK.

Chapter 3

Units of Measurement

In the first chapter, we saw that km, m, cm, and mm are units of length. We know that

$$1\,\mathrm{km} = 1000\,\mathrm{m}$$

but of course, $1 \neq 1000$. What's going on here?

In the last chapter, we saw two units of measurement for angles: the turn and the degree. We learned that one turn is equivalent to $360°$. But from common sense, we know that $1 \neq 360$. How can it be?

The answer is quite simple: while km and m are both units of length, they are not the same unit of length. Similarly, although both the turn and the degree are units of angle measure, they are not the same unit of angle measure.

TK.

Chapter 4

Length and Area

In previous units, we looked at how to measure angles. Angles are a way of measuring the size of the gap between two intersecting rays. We will now look at another important geometric quantity: length.

We have all encountered length in everyday life. Some things are longer than others. Geometrically, length is the measure of the size of a line segment. Figure 4.1 depicts two line segments with different lengths. We can see from the diagram that line segment B is longer than line segment A, but how can we measure that? Our goal is to attach a number to both line segments so we can say more precisely how long they are.

TK.

4.1 Units of Length

TK.

4.2 Distance

In past sections, we have looked at how to measure the length of particular line segments. Another way to think of the length of a line segment is that it is the distance between the two ends. TK.

4.3 Area of a Rectangle

A rectangle is a quadrilateral (four-sided polygon) with four right angles. See Figure 4.2 for an example. The area of a rectangle is given by the equation

$$A = \ell \times w \tag{4.1}$$

where ℓ represents the length and w the width of the rectangle.

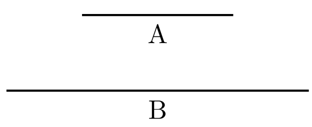

Figure 4.1: Two line segments of different lengths; B is longer than A

Figure 4.2: A rectangle

Example 4a. **Area of a Rectangle**

Find the area of each rectangle.

Width	Length	Area
10 m	5 m	
8 m	12 m	

Solution: We use the formula for the area of a rectangle to obtain:

Width	Length	Area
10 m	5 m	$\boxed{50\,\text{m}^2}$
8 m	12 m	$\boxed{96\,\text{m}^2}$

A special case of a rectangle is a square, where the four sides are all equal and meet at right angles. TK.

4.4 Area of a Triangle

A triangle is a polygon with exactly three sides. We can classify triangles by the measure of their largest angle. Figure 4.3 depicts a triangle with an obtuse angle. Triangles with an obtuse angle are called obtuse triangles. Triangles cannot have more than one obtuse angle. Why? The sum of the interior angles of a triangle is 180°, and two obtuse angles already sum to more than 180°.

Figure 4.4 depicts a triangle with a right angle. Triangles with a right angle are called right triangles. Could a triangle have two right angles? If it did, then the third angle would have to be 0°. But two lines intersecting at 0° would have to be coincident, so our triangle will degenerate into a line segment. Generally, we don't call line segments triangles. Similarly, a triangle couldn't have a straight or reflex angle — not even a single one.

Finally, figure 4.5 depicts a triangle with three acute angles. This is possible. In fact, many triangles you draw will have three acute angles. Triangles with only acute angles are called acute triangles.

An altitude of a triangle is a line segment drawn from a vertex to the opposite side, which intersects the opposite side at a right angle. Each triangle has three altitudes. When we measure a triangle, we usually pick one side of the triangle to call the base. The length of the altitude corresponding to that side is called the "height" of the triangle. Often, the base is drawn at the bottom of the

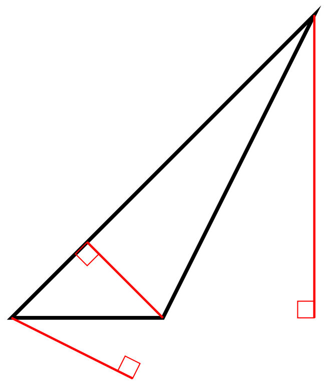

Figure 4.3: A triangle with an obtuse angle

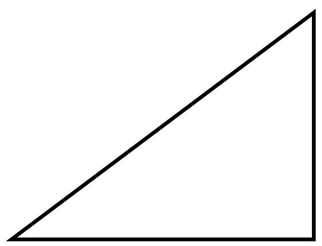

Figure 4.4: A triangle with a right angle

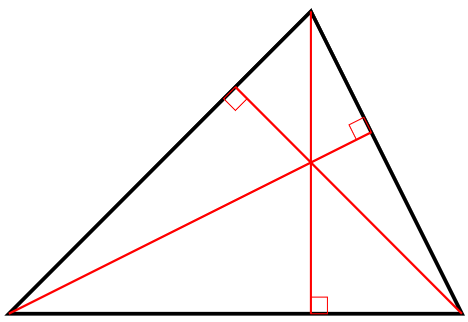

Figure 4.5: A triangle with three acute angles

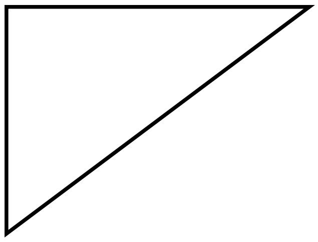

Figure 4.6: A triangle with a right angle, rotated 180°

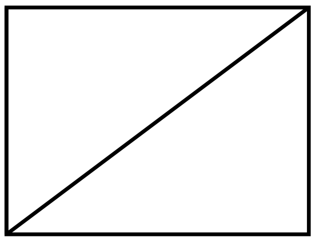

Figure 4.7: Two right triangles put together to form a rectangle

triangle, but this is not required. After all, we can rotate a triangle without changing its shape.

Let's try to figure out a formula for the area of a triangle. The triangle that looks simplest is the right triangle, so let's look at that first. Make the base one of the sides incident to the right angle. Then, the height is the other side. Consider the triangle drawn in Figure 4.4 first. Let's rotate it 180°, to get Figure 4.6.

This triangle can fit together with the original triangle like puzzle pieces. The resulting shape is a rectangle, as seen in Figure 4.7. Since both these triangles are the same size, that means that the area of one of them is half the area of the rectangle. From this, we come to our first conclusion: for a right triangle, the area is one half the product of the height and the base.

For an acute triangle, make any side the base and draw the height. We now put the triangle into a box, like in Figure 4.8. The box is split into two smaller rectangles by the triangle's altitude. Each rectangle has half contained within the triangle and half not contained. So the triangle is, once again, half the area of the rectangle. The rectangle's area is the product of the height and the base, so the area of the triangle is one half the product of the height and the base. This is the same result as we obtained with the right triangle.

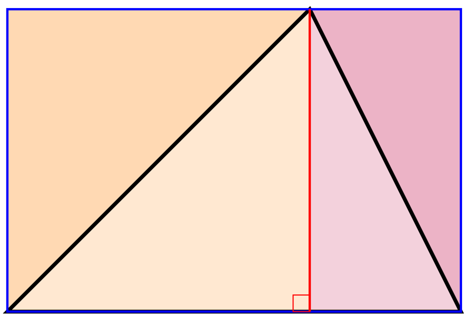

Figure 4.8: An acute triangle with a rectangle drawn around it

In fact, even obtuse angles satisfy the same formula. (That is a trickier problem to solve, but can be shown the same way through dividing up triangles and rectangles.) The area of a triangle is always given by

$$A = \frac{1}{2} \times b \times h \tag{4.2}$$

where b is the base and h is the height of the triangle.

Example 4b. **Area of a Triangle**

Complete the table for each triangle.

Base	Height	Area
15 m	8 m	
	14 m	49 m^2
20 m		100 m^2

Solution: We use the formula for the area of a triangle to obtain:

Base	Height	Area
15 m	8 m	$\boxed{60\,\text{m}^2}$
$\boxed{7\,\text{m}}$	14 m	49 m^2
20 m	$\boxed{10\,\text{m}}$	100 m^2

4.5 Area of a Parallelogram

A parallelogram is a quadrilateral with two sets of parallel sides. A parallelogram can be split into two triangles, and from that we can obtain the formula for the area of a parallelogram:

$$A = b \times h \tag{4.3}$$

where b represents the base and h represents the height.

Example 4c. **Area of a Parallelogram**

Base	Height	Area
15 m	4 m	
20 m		60 m^2

Solution: We use the formula for the area of a parallelogram to obtain:

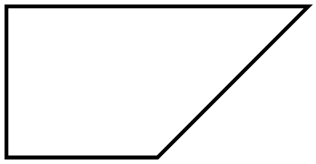

Figure 4.9: A trapezoid that is not a parallelogram; the top and bottom sides are parallel

Base	Height	Area
15 m	4 m	$\boxed{60\,\mathrm{m}^2}$
20 m	$\boxed{3\,\mathrm{m}}$	$60\,\mathrm{m}^2$

TK.

4.6 Area of a Trapezoid

What is a trapezoid? Trapezoids are shapes that have at least one set of parallel sides. Since parallelograms have two sets of parallel sides, and two is at least one, that means that all parallelograms are trapezoids.

Example 4d. **Hierarchy of Quadrilaterals**

Are all trapezoids parallelograms? If not, draw a trapezoid that is not a parallelogram.

Solution: $\boxed{\text{No.}}$ Figure 4.9 depicts one.

TK.

Chapter 5

Geometric Theorems

5.1 Squares and Square Roots

Before continuing in geometry, we will talk about some arithmetic. Recall that the area of a rectangle is given by

$$A = \ell \times w$$

Then, what is the area of a square? All squares are rectangles: they have two sets of parallel sides which meet at right angles. In fact, we can use the formula for the area of a rectangle. This is just a special case where the length and the width happen to be the same number. Figure 5.1 depicts a square with side length s. Then the width and the length of this square are both s. That is,

$$\ell = w = s$$

Now to calculate the area of the square, we can use the formula for the area of a rectangle:

$$A = \ell \times w = s \times s$$

and so the result is simply the side length multiplied with itself.

It turns out that multiplying numbers by themselves is common enough that we give this operation its own name and notation. If n is a number, then we denote by n^2 (pronounced "n squared") the quantity $n \times n$. We are simply defining this notation, which will save us some writing.

Using this notation, the formula for the area of a square is given by

$$A = s^2 \tag{5.1}$$

where s is the side length of the square.

We must be careful when squaring quantities with units, because squaring is a multiplication, and so we must multiply the unit with itself too. This is where the notation m^2 mentioned in the last unit comes from. TK.

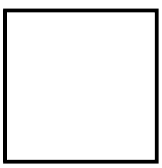

Figure 5.1: A square with side length s

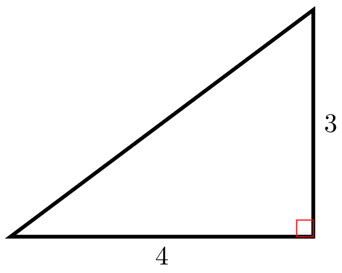

Figure 5.2: A triangle with a right angle

5.2 Right Triangles

Recall that a right triangle is a triangle with one right angle. Consider the right triangle in Figure 5.2.

We call the two sides of a right triangle incident to the right angle the legs. In Figure 5.2, the lengths of the two legs are 3 and 4. The side opposite to the right angle is called the "hypotenuse". From the lengths of the legs, can we figure out the length of the hypotenuse?

TK.

5.3 Using Variables

Above we derived the length of the hypotenuse for two specific sets of leg lengths. Recall that we have formulas for the area of several kinds of shapes. For instance, the formula for the area of a triangle is

$$A = \frac{1}{2} b \times h$$

where the letters b and h represent the actual lengths of the base and altitude.

Can we find a formula for the length of the hypotenuse, using letters a and b to represent the leg lengths? If we could do this, then we can find the length of the hypotenuse for any right triangle, without needing to go through the tedious process we did above.

TK.

5.4 The Pythagorean Theorem

In the section above, we derived a formula for the length of the hypotenuse using the lengths of the two legs. If the lengths of the two legs are a and b, and the length of the hypotenuse is c, then we saw that

$$c^2 = a^2 + b^2 \tag{5.2}$$

Equation 5.2 is known as the Pythagorean Theorem. This is a very useful theorem for many geometry problems.

TK.

Glossary

acute angle an angle measuring less than a right angle (90° or a quarter of a turn); such angles are typically characterized as being sharp. 6

angle the figure formed by two rays, called the sides of the angle, sharing a common endpoint, called the vertex of the angle (Wikipedia). 5

coincident a description of two objects which occupy exactly the same space. 3

endpoint an extreme point of a line segment or ray; line segments have two endpoints whereas rays have just one. 3

line a straight one-dimensional object that extends forever in both directions. 2

line segment a straight one-dimensional object terminated at both ends. 2

obtuse angle an angle measuring more than a right angle (90° or a quarter of a turn) but less than a straight angle (180° or a half of a turn). 6

parallel a description of two lines that never intersect at any point. 3

plane a two-dimensional flat surface. 2

plane geometry the study of figures on a plane (a two-dimensional flat surface). 2

ray a straight one-dimensional object terminated at one end and extending forever in the other direction. 2

reflex angle an angle measuring more than 180° or a half of a turn, but less than 360° or a full turn. 7

right angle an angle measuring 90° or a quarter of a turn; angle between two rays intersecting in an L shape. 5

right triangle an triangle with one 90° (right) angle. 13, 19

straight angle an angle measuring 180° or a half of a turn; angle between two rays in opposite directions. 6

List of Figures